VIBRATIONS OF AN OSCILLATOR WITH RESIDUAL CREEP

(KOLEBANIIA OSTSILLIATORA,OBLADAIUSHCHEGO NASLEDSTVENNOI POLZUCHEST'IU)

PMM Vol. 30, No. 3, 1966, pp. 584-589

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(Received May 10, 1965)

The authors consider the vibrations of an oscillator with elasto-residual linear and slightly non-linear characteristics. The investigations using the operator methods, are based on the residual creep theory and and low-order exponentials are adopted as the relaxation residuals [1].

1. The relationship between the stress σ and deformation ε in the residual theory of creep, is given by Rabotnov [2], in the form

$$\varphi(\varepsilon) = [1 + \varkappa \vartheta_\alpha^*(\beta_1)]\sigma \tag{1.1}$$

$$\vartheta_\alpha^*(\beta) f(t) \equiv \int_0^t \vartheta_\alpha(\beta;\ t-\tau) f(\tau)\, d\tau, \qquad \vartheta_\alpha(\beta;\ t) = \sum_{n=0}^{\infty} \frac{\beta^n t^{n(1+\alpha)+\alpha}}{\Gamma[(n+1)(1+\alpha)]} \quad (-1 < \alpha < 0)$$

The creep parameters $\varkappa$ and β_1 and the exact form of function φ are normally determined experimentally. Utilising the properties of the operators $\vartheta_\alpha^*(\beta)$ [1], and using a particular form of the $\varphi(\varepsilon)$ $E_0 = \varepsilon[1-\gamma q(\varepsilon)]$, we can express (1.1) in the form

$$\sigma = E_t\, \varepsilon\, [1 - \gamma q(\varepsilon)], \qquad E_t = E_0\,[1 - \varkappa \vartheta_\alpha^*(\beta)], \qquad \beta = \beta_1 - \varkappa \tag{1.2}$$

Here, E_t is the modulus of elasticity in the operator form, E_0 is the instantaneous modulus of elasticity and $\varkappa$ and β are the relaxation parameters.

Substituting by Volterra's principle, the instantaneous modulus of elasticity with its operator analogue in the equation describing free vibrations of an elastic oscillator we have, in the linear case,

$$x'' + 2hx' + \omega_0^2\,[1 - \varkappa \vartheta_\alpha^*(\beta)]\, x = 0 \tag{1.3}$$

where x is the displacement while h and ω_0 are constants. This equation of motion corresponds to the case when $\gamma = 0$ in (1.2).

Use in Equation (1.3) of a simple exponential ($\alpha = 0$) as the relaxation kernel, leads to a qualitatively correct result [3]. In this case, the characteristic equation, corresponding to a third order differential equation to which Equation (1.3) is reduced, assumes the form

$$k^3 + (2h - \beta) k^2 + (\omega_0^2 - 2h\beta) k - \omega_0^2 (\varkappa + \beta) = 0 \tag{1.4}$$

when $\omega_0^2 \geqslant 2h\beta + {}^1/_3 (2h - \beta)^2$ and since the relaxation parameters satisfy [4] the inequalities $\varkappa > 0$, $\varkappa + \beta < 0$, the roots of the governing equation include one real root k_1 and two complex roots. These roots satisfy the conditions

$$-{}^2/_3\, h + \beta < k_1 < 0, \qquad -{}^1/_2 (2h - \beta) < \mathrm{Re}\, k_{2\,3} < -{}^2/_3\, h \tag{1.5}$$

In view of this, the solution of Equation (1.3)

$$x = A_1 e^{k_1 t} + A_2 e^{k_2 t} + A_3 e^{k_3 t} \tag{1.6}$$

describes a decaying process.

However, the use of simple exponentials as the residual kernels leads to quantitative results which are unsatisfactory. More reliable results can be obtained only by means of a more accurate process of relaxation utilising weakly-singular residuals [5, 3 and 1]. It is more convenient to use the residual kernel proposed by Rabotnov — a small-order expoential $\vartheta_\alpha (\beta; t)$, whose properties are discussed below.

2. Laplace's transform of the function $\vartheta_\alpha (\beta; t)$ is of the form [6]

$$L\, [\vartheta_\alpha (\beta; t)] = (p^r - \beta)^{-1} \qquad (r = 1 + \alpha) \tag{2.1}$$

where L denotes the operator of the Laplace transform.

The index α usually determined during the construction of experimental creep or relaxation curves can normally be quite accurately expressed by a proper fraction. Let $r = a/c$, where a and c are integers. Expression (2.1) can also be used in many cases, to find the relationship between the function $\vartheta_\alpha (\beta; t)$ and the tabulated incomplete gamma-function $\Gamma (m; x)$ together with the probability integral $\Phi (x)$

$$\Gamma (m; \; x) = \int_0^x s^{m-1} e^{-s} ds, \qquad \Phi (x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds$$

Thus, the transformation

$$L\, [\vartheta_{-1/2} (\beta; \; t)] = \frac{1}{\sqrt{p} - \beta} = \frac{1}{\sqrt{p}} + \frac{\beta^2}{\sqrt{p}\,(p - \beta^2)} + \frac{\beta}{p - \beta^2}$$

results in the relationship quoted in [1]

$$\vartheta_{-1/2} (\beta; \; t) = \frac{1}{\sqrt{\pi t}} + \beta e^{\beta^2 t} [1 + \Phi (\beta \sqrt{t})] \tag{2.2}$$

Similarly, we have

$$\vartheta_{-1/2}^{*} (\beta) \cdot 1 = -\frac{1}{\beta} \{1 - e^{\beta^2 t} [1 + \Phi (\beta \sqrt{t})]\} \tag{2.3}$$

$$\vartheta_{-2/3} (\beta; \; t) = \frac{t^{-2/3}}{\Gamma (^1/_3)} + \frac{\beta t^{-1/3}}{\Gamma (^2/_3)} + \beta^2 e^{\beta^3 t} \left[1 + \frac{\Gamma (^1/_3; \; \beta^3 t)}{\Gamma (^1/_3)} + \frac{\Gamma (^2/_3; \; \beta^3 t)}{\Gamma (^2/_3)}\right] \tag{2.4}$$

$$\vartheta_{-2/3}^{*} (\beta) \cdot 1 = -\frac{1}{\beta} \left\{1 - e^{\beta^3 t} \left[1 + \frac{\Gamma (^1/_3; \; \beta^3 t)}{\Gamma (^1/_3)} + \frac{\Gamma (^2/_3; \; \beta^3 t)}{\Gamma (^2/_3)}\right]\right\} \tag{2.5}$$

$$\vartheta_{-1/2}^{*} (\beta)\, e^{\lambda t} = \frac{\beta}{\beta^2 - \lambda}\, e^{\beta^2 t} [1 + \Phi (\beta \sqrt{t})] - \frac{e^{\lambda t}}{\beta^2 - \lambda} [\sqrt{\lambda}\, \Phi (\sqrt{\lambda t}) + \beta] \tag{2.6}$$

For the case when $-1 < \alpha < 0$ approximations can be used, which hold for any β (which in general are complex)

$$\vartheta_\alpha(\beta;\ t) \sim -\sum_{n=0}^{\infty} \frac{t^{-nr-1}}{\beta^{n+1}\Gamma(-nr)} + \operatorname{res} \frac{e^{pt}}{p^r-\beta} \tag{2.7}$$

Here and in the following, the residues are computed at the poles distributed on the branch of the complex variable p where $-\pi \leqslant \arg p < \pi$ (the branch point at $p=0$ is excluded). The error is estimated from the formula

$$|r_N(t)| \leqslant \frac{\Gamma(rN+r+1)}{\pi g\,|\beta|^{N+1}}\, t^{-rN-r-1} \tag{2.8}$$

$$g = \begin{cases} |\beta| \sin\theta & (0 < \theta \leqslant {}^1\!/_2\pi), \\ |\beta| & ({}^1\!/_2\pi \leqslant \theta), \end{cases} \qquad \theta = |\,\pi r - |\arg\beta|\,|$$

$$\vartheta_\alpha^*(\beta)\cdot 1 \sim -\sum_{n=0}^{\infty} \frac{t^{-nr}}{\beta^{n+1}\,\Gamma(1-nr)} + \operatorname{res}\frac{e^{pt}}{p\,(p^r-\beta)} \tag{2.9}$$

$$|r_N(t)| \leqslant \frac{\Gamma(rN+r)}{\pi g\,|\beta|^{N+1}}\, t^{-rN-r} \tag{2.10}$$

For a real β, Equations (2.7) and (2.9) coincide with those given in [7], and the estimate of error is obtained in the same way as in [6].

$$L[\vartheta_\alpha^*(\beta)\,e^{\lambda t}] = \frac{1}{(p^r-\beta)(p-\lambda)} = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \beta^{-l-1}\lambda^{-m-1} p^{\frac{al}{c}+m} = \sum_{k=0}^{\infty} b_k p^{\frac{k}{c}}$$

Here

$$r = \frac{a}{c}, \qquad b_k = \sum_{m=0}^{M} \beta^{\frac{mc-k}{a}-1} \lambda^{-m-1}\delta(mc+al-k)$$

$$\delta(mc+al-k) = \begin{cases} 0 & (mc+al \neq k) \\ 1 & (mc+al = k) \end{cases}$$

By theorem 11 of [8] (p. 218), we have

$$\vartheta_\alpha^*(\beta)\,e^{\lambda t} \sim \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(-k/c)}\, t^{-\frac{k}{c}-1} + \sum \operatorname{res} \frac{e^{pt}}{(p-\lambda)(p^r-\beta)} \tag{2.11}$$

For an arbitrary complex λ, or a real $\lambda > 0$, the error is given by the formula

$$\begin{aligned} |r_K(t)| \leqslant \frac{1}{\pi g f}\Big[& \frac{\Gamma(rL+r+1)}{|\beta|^{L+1}}\, t^{-rL-r-1} + \frac{\Gamma(M+2)}{|\lambda|^{M+1}}\, t^{-M-2} + \\ & + \frac{\Gamma(rL+M+r+2)}{|\beta|^{L+1}|\lambda|^{M+1}}\, t^{-rL-M-r-2} \end{aligned} \tag{2.12}$$

Here

$$f = \begin{cases} |\lambda|\sin\psi & (0 < \psi \leqslant {}^1\!/_2\pi), \\ |\lambda| & ({}^1\!/_2\pi \leqslant \psi), \end{cases} \qquad \psi = |\,\pi - |\arg\lambda|\,|, \qquad K = Mc + aL$$

3. Let us return to Equation (1.4). Using the Laplace transformation we obtain, for $h=0$

$$L[x] = \frac{px(0) + x^{\cdot}(0)}{p^2 + \omega_0^2 [1 - \varkappa (p^r - \beta)^{-1}]} \tag{3.1}$$

Substitution $p = z^c$ $(r = a/c)$ allows us to rationalise the expression (3.1) and to reduce it to elementary fractions

$$L[x] = \sum_k A_k [p^{1/c} - \beta_k]^{-1} \tag{3.2}$$

By (2.1), the solution of the equation of free vibrations of an oscillator with residual creep, has the form

$$x = \sum_k A_k \vartheta_{1/c-1}(\beta_k;\ t) \tag{3.3}$$

The singularity of the function $\vartheta_\alpha(\beta; t)$ at the initial moment $t = 0$, does not result in a singularity in the solution of Equation (3.3). This can easily be checked by means of limit theorems of the operator calculus. Approximation (2.7) allows the solution (3.3) to be brought to a numerical result. Use of the formula (2.7), brings into the solution for (3.3) terms in the form $A \exp \beta_k^{ct}$ corresponding to those roots $p_k = \beta_k^c$ of the denominator of (3.1), for which $|\arg \beta_k| < \pi/c$.

The complex root p_k of the equation

$$p^2 + \omega_0^2 = \frac{\varkappa \omega_0^2}{p^r - \beta} \tag{3.4}$$

can be expressed in the exponential form $p_k = R_k e^{i\varphi_k}$, $0 < |\varphi_k| < \pi$. By (3.4), we have

$$R_k^2 e^{2i\varphi_k} + \omega_0^2 = \varkappa\omega_0^2 \,|\, R_k^r e^{ir\varphi_k} - \beta \,|^{-2} (R_k^r e^{-ir\varphi_k} - \beta)$$

Equating the imaginary parts on either side of the equation, we find that at $\varkappa > 0$, $\sin 2\varphi_k$ has the sign opposite to that of $\sin r\varphi_k$. This is possible if $1/2\,\pi < |\varphi_k| < \pi$. Consequently the real part of the complex root $p_k = \beta_k^c$ is negative. Since $\varkappa + \beta < 0$ $\varkappa > 0$, the real roots of Equation (3.4) can only be negative, and this ensures an asymptotic stability of the solution of (3.3).

Assuming that creep has little effect on the frequency of oscillations, we shall take $p_{k0} = i\omega_0$ which corresponds to the case $\varkappa = 0$. Substituting the value of p_{k0} into the right-hand side of Equation (3.4) we obtain the first approximation

$$p_{k1} = i\omega_0 \left(1 - \frac{\varkappa}{(i\omega_0)^r - \beta}\right)^{1/2}$$

Since $|\beta| > \varkappa$ and $\beta < 0$, we have $\arg[(i\omega_0)^r - \beta] < 1/2\,\pi r$ and $\varkappa\,|\,(i\omega_0)^r - \beta\,|^{-1} < 1$. From this it follows that $|1 - \varkappa[(i\omega_0)^r - \beta]^{-1}| < 1$ and, thus, the frequency of free oscillations with an allowance for the residual properties of the material is $\omega = \operatorname{Im} p_k < \omega_0$.

In the case of forced oscillations, we have

$$x = y + Y, \qquad L[x] = L[y] + \left[p^2 + \omega_0^2\left(1 - \frac{\varkappa}{p^r - \beta}\right)\right]^{-1} L[f(t)]$$

where y are the free oscillations in the form (3.3); Y are the forced oscillations, and $f(t)$ is the exciting force.

When the initial conditions are $x(0) = 0$, $x^{\cdot}(0) \neq 0$, using the convolution theorem and taking into account (3.2), we obtain

$$x = \sum_k A_k x^{\cdot}(0)\,\partial_{1/c-1}(\beta_k;\ t) + \sum_k A_k \partial^{*}_{1/c-1}(\beta_k) f(t) \tag{3.5}$$

In the case $f(t) = A \sin \omega_1 t$, the forced oscillations Y contain the expressions of the form

$$\begin{gathered}\partial^{*}_{1/c-1}(\beta_k)\sin\omega_1 t \sim \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(-k/c)}\, t^{-\frac{k}{c}-1} + \frac{e^{i\omega_1 t}}{2i\,[(i\omega_1)^{1/c}-\beta_k]} - \\ - \frac{e^{-i\omega_1 t}}{2i\,[(i\omega_1)^{1/c}-\beta_k]} + \frac{c\beta_k{}^{c-1}\omega_1}{\beta_k{}^{2c}+\omega_1{}^2}\exp \beta_k{}^c t\end{gathered} \tag{3.6}$$

This approximation is derived in the same way as (2.11). Only the second and the third term do not decay with time, and because of this, equation (3.5) will, after large enough interval of time, assume the form

$$x = \frac{1}{2i}\left[\omega_0{}^2 - \omega_1{}^2 - \frac{\varkappa\omega_0{}^2}{(i\omega_1)^r - \beta}\right]^{-1} e^{i\omega_1 t} - \frac{1}{2i}\left[\omega_0{}^2 - \omega_1{}^2 - \frac{\varkappa\omega_0{}^2}{(-i\omega_1)^r - \beta}\right]^{-1} e^{-i\omega_1 t} \tag{3.7}$$

Since the roots p_k of Equation (3.4) do not include pure imaginary roots, $p_k \neq i\omega_1$ which means that the harmonic oscillations in (3.7) are of finite amplitude. If it is assumed that under resonant conditions η_1 does not differ significantly from ω_0, then the approximate condition of resonance can be given by

$$\omega_1{}^2 = \omega_0{}^2\,[1 - \varkappa\,|\,(i\omega_0)^r - \beta\,|^{-2}\,(\omega_0{}^r \cos {}^1\!/_2\,\pi r - \beta)] \tag{3.8}$$

In view of the fact that the resonant frequency can have only one value, one may expect that not more than two of the harmonic components of (3.5) correspond to the pair of conjugate roots β_k which are, under resonant conditions, substantially larger than the rest. Then, in view of (3.6), with

$$\omega_1{}^{1/c} = u_k \cos\frac{\pi}{2c} + v_k \sin\frac{\pi}{2c} \tag{3.9}$$

the maximum amplitude of those components will reach

$$\left| u_k \sin\frac{\pi}{2c} - v_k \cos\frac{\pi}{2c}\right|^{-1} \tag{3.10}$$

Here, u_k and v_k are, respectively, the real and the imaginary part of root β_k respectively for which Equation (3.9) is satisfied.

As was shown above, ${}^1\!/_2\,\pi / c < |\arg \beta_k| \leqslant \pi / c$. From (3.10) it follows, that as $|\arg \beta_k|$ tends to $\frac{1}{2}\pi/c$ the maximum amplitude increases without limit.

The limiting condition of resonance occurs only when $|\arg\beta_k| = {}^1\!/_2\,\pi/c$, which corresponds to total absence of creep ($\varkappa = 0$).

The equation for free vibrations of an oscillator under the conditions corresponding to the non-linearity of Equation (1.2), is of the form

$$x^{\cdot\cdot} + \omega_0{}^2\,[1 - \varkappa\partial_\alpha{}^*(\beta)]\,[1 - \gamma q(x)]\,x = 0 \tag{3.11}$$

Neglecting the terms of second order and assuming that in this particular case

$q(x) = x^2$, we can rewrite (3.11) in the form

$$x^{\cdot\cdot} + \omega_0^2 x - \varkappa\omega_0^2 \vartheta_\alpha^* (\beta)\, x = \gamma\omega_0^2 x^3 \tag{3.12}$$

Let us now use the method of successive approximations. As the zero th approximation, we shall take the solution (3.3) of Equation (3.12) at $\gamma = 0$. As stated above, the use of an exponential kernel leads to a qualitatively correct solution in the form of a linear combination of three exponents. Indeed, the zero th approximation x_0 can, with sufficient accuracy, be represented by exponential expressions.

Substituting x_0 into the right-hand side of Equation (3.12), we find

$$L[x] = L[x_0] + \sum_l \frac{B_l \gamma\omega_0^2}{(p - \lambda_l)[p^2 + \omega_0^2 - \varkappa\omega_0^2 (p^r - \beta)^{-1}]}$$

In particular, when $x(0) = 0,\ x^{\cdot}(0) \neq 0$, we obtain for the first approximation

$$x_1 = \sum_k A_k x^{\cdot}(0)\, \vartheta_{1/c-1}(\beta_k;\ t) + \sum_l \sum_k A_k B_l \gamma\omega_0^2 \vartheta^*_{1/c-1}(\beta_k)\, e^{\lambda_l t} \tag{3.13}$$

The above approximations can be used to obtain a numerical solution. Expressing x_1 exponentially and substituting them into the right-hand side of Equation (3.12) we can obtain the second approximation, etc.

4. As an illustration, let us consider the following examples. The relaxation curve for armature iron [9] at $T = 500°C$ under initial stress $\sigma_0 = 10.78 \times 10^7$ n/m² can, with a 7% accuracy be described by (1.2) with $\gamma = 0$, $\alpha = -0.5$, $\beta = -0.0183$ sec$^{-0.5}$ and $\varkappa = 0.0146$ sec$^{-0.5}$.

The problem of oscillations of a load $W = 49 \times 10^3$ newtons placed at mid-span of such beam of length $l = 4$ m, with a moment of inertia $I = 2.45 \times 10^{-6}$ m⁴ and $E_0 = 1{,}764 \times 10^{11}$ h/m², reduces to the solution of the equations (ignoring the mass of the beam)

$$x^{\cdot\cdot} + 649.7x - 9.473\vartheta_{-1/2}^* (-0.0183)\, x = 0 \tag{4.1}$$

According to (3.1) to (3.3), when $x(0) = 0,\ x^{\cdot}(0) \neq 0$, we have

$$x = x^{\cdot}(0) \sum_{k=0}^{4} A_k \vartheta_{-1/2}(\beta_k;\ t) \tag{4.2}$$

Here

$$A_0 = 2.24 \cdot 10^{-5},\quad A_1 = (1.368 - 1.374i)\, 10^{-3},\quad A_2 = (1.368 + 1.374i)\, 10^{-3}$$
$$A_3 = (-1.379 - 1.374i)\, 10^{-3},\quad A_4 = (-1.379 + 1.374i)\, 10^{-3}$$
$$\beta_0 = -3.749 \cdot 10^{-3},\quad \beta_1 = -3.574 + 3.570i,\quad \beta_2 = -3.574 - 3.570i$$
$$\beta_3 = 3.566 + 3.570i,\quad \beta_4 = 3.566 - 3.570i$$

Using (2,2) and (2.7) we obtain

$$x = 0.03928\, x^{\cdot}(0)\, e^{-0.0285 t} \sin 25.46\, t \tag{4.3}$$

The approximation used above produce an error in the amplitude not exceeding 0.5%.

If exponential residuals are used to obtain the relaxation curve (which can be done by neglecting the initial singularity in the curve, the obtained parameters are $\varkappa = 0.94 \times 10^{-6}$ sec^{-1}, $\beta = -2.6 \times 10^{-6}$ sec^{-1} and $\alpha = 0$. The solution of Equation (4.1) will in this case take the form

$$x = 0.0392 e^{x^{\cdot}(0) - 0.47 \cdot 10^{-6}\, t} \sin 25.49t \tag{4.4}$$

Thus, the use of exponential residual leads to the lowering of the logarithmic decrement by the factor of approximately 6×10^4.

By completing the programme of successive approximations which was described above for equation (3.12), under the initial conditions $x(0) = 0$, $x^{\cdot}(0) \neq 0$, we shall obtain the first approximation

$$x_1 = 0.03928 x^{\cdot}(0)\, e^{-0.0235\, t} \sin 25.46t + 5.67 \times 10^{-4}\, x^{\cdot 3}(0)\, \gamma e^{-0.0855\, t} (0.14 \sin 25.46\, t - \\ - t \cos 25.46t) \tag{4.5}$$

The need for a small non linear correction with a given $x^{\cdot}(0)$ imposes some real limitations upon the parameter γ.

The drop in the frequency of free oscillations, which is noted when creep is taken into account, becomes particularly pronounced when parameters β and $\varkappa$ are sufficiently large. For rubber-like material with rheological parameters $\alpha = -1/3$, $\beta = -1.95\ \mathrm{sec}^{-2/3}$ and $\varkappa = 0.75\ \mathrm{sec}^{-2/3}$, we obtain for $\omega_0 = 10\ \mathrm{sec}^{-1}$, $x(0) = 0$, and $0 \neq (0)\, .x$, the following equation of free oscillations

$$x^{\cdot\cdot} + 100x - 75\mathcal{E}_{-1/3}^{*}(-1.95)\, x = 0 \tag{4.6}$$

After only 2 seconds, the amplitude of these oscillations can be expressed with 0.25% accuracy, in the form

$$x = 0.14\, x^{\cdot}(0)\, e^{-0.44\, t} \sin 9.52\, t \tag{4.7}$$

Thus, the frequency diminishes by 4.8%.

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Translated by S.K.